STABILITY OF A CURVILINEAR MOTION OF A VEHICLE ON WHEELS WITH PNEUMATIC TIRES

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A general approach towards the study of stability of motion of a vehicle on wheels with pneumatic tires used in [1] for the case of a rectilinear, unperturbed motion, is extended to the case of a curvilinear motion along a path of sufficiently small curvature.

1. Statement of the problem. We assume that the conditions under which the theory of rolling of a pneumatic tire wheel developed by Keldysh in [2] is valid, hold in the present case. According to this theory the rolling of the wheel takes place without slipping, while the deformation of the tire is small and characterized by three parameters: the quantity ξ describing the lateral displacement of the center of the area of contact relative to the trace of the middle plane of the wheel on the road surface, the angle χ of inclination of the middle plane of the wheel and the angle φ of torsion of the tire. The fact that the tire undergoes small deformation and the condition of rolling without slipping, impose definite restrictions on the class of motions under consideration. In particular, the path curvature must be small and the velocity of motion must not become excessive.

Let us denote by $q_1, q_2, \ldots, q_n$ the generalized coordinates of a vehicle on m pneumatic tire wheels and introduce quantities determining the position of the ith wheel $(i = 1, 2, \ldots, m)$. Let x_i, y_i be the Cartesian coordinates of the point K_i of intersection of the steepest line passed along the middle plane of the wheel through its center with the plane of the road, θ_i the angle formed by the trace of the middle plane of the wheel on the road and the Ox-axis of the fixed $Oxyz$ coordinate system the xOy-plane of which coincides with the plane of the road while the Oz-axis points upwards, and χ_i the angle between the Oz-axis and the mean plane of the wheel. The coordinates x_i, y_i, θ_i and χ_i introduced here are known functions of the generalized coordinates $q_1, q_2, \ldots, q_n$.

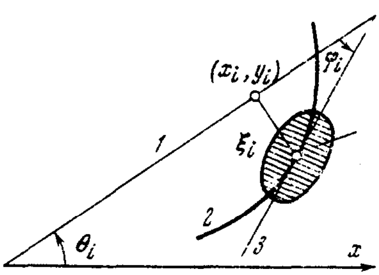

Fig. 1.

At first we assume that the motion of the vehicle is given. This means that x_i, y_i, θ_i, and χ_i are known functions of time. Then by the Keldysh theory the deformation of the pneumatic tire can be found at any instant on the basis of the following conditions:

1) the tangent to the line of rolling of the tire coincides with the axis of the area of contact and

2) the curvature of the line of rolling is determined uniquely by the deformation of the tire.

In accordance with the notation of Fig. 1 (*1* is the trace of the middle plane of the wheel, *2* is the line of rolling, *3* is the axis of the area of contact and *4* is its center) these conditions lead to the relations

$$dx_i^* \sin(\theta_i + \varphi_i) - dy_i^* \cos(\theta_i + \varphi_i) = 0 \qquad (1.1)$$

$$R_i^{-1} = \alpha_i \xi_i - \beta_i \varphi_i - \gamma_i \chi_i \qquad (1.2)$$

Here x_i^*, y_i^* are the coordinates of the center of the area of contact connected with x_i, y_i respectively, by

$$x_i^* = x_i + \xi_i \sin\theta_i, \qquad y_i^* = y_i - \xi_i \cos\theta_i \qquad (1.3)$$

where R_i is the radius of curvature of the line of rolling and α_i, β_i and γ_i are constant coefficients called the kinematic parameters of the ith wheel determined by experiment. Using (1.3) and neglecting the terms of the second and higher order of smallness we obtain, in place of (1.1),

$$dx_i \sin(\theta_i + \varphi_i) - dy_i \cos(\theta_i + \varphi_i) + d\xi_i = 0 \qquad (1.4)$$

By definition, the curvature of the plane curve $R_i^{-1} = d(\theta_i + \varphi_i) / ds_i$, where ds_i is the arc element of the line of rolling of the ith tire. Inserting this into (1.2) we obtain

$$d\theta_i + d\varphi_i - ds_i (\alpha_i \xi_i - \beta_i \varphi_i - \gamma_i \chi_i) = 0 \qquad (1.5)$$

Equations (1.4) and (1.5) represent the required relations from which the deformations ξ_i and φ_i can be found provided that the motion of the tire wheel is known. Having found the deformations we can now determine the forces acting on the ith wheel. According to the Keldysh theory these forces are equivalent to the transverse force F_i applied at the point K_i, the moment M_{θ_i} relative to the vertical axis and the moment M_{χ_i} relative to the horizontal axis parallel to the middle plane of the wheel. Moreover we have

$$F_i = a_i \xi_i + \sigma_i N_i \chi_i, \qquad M_{\theta_i} = b_i \varphi_i, \quad M_{\chi_i} = -\sigma_i N_i \xi_i - \rho_i N_i \chi_i \qquad (1.6)$$

where N_i denotes the load on the ith tire wheel, while a_i, b_i, σ_i and ρ_i are constant coefficients determined by experiment.

2. Kinematic and dynamic equations of motion. Let us divide (1.4) and (1.5) by dt. Eliminating $s_i^{\cdot}$ by means of the relation

$$s_i^{\cdot} = x^{*\cdot}_i \cos(\theta_i + \varphi_i) + y_i^{*\cdot} \sin(\theta_i + \varphi_i) =$$
$$= x_i^{\cdot} \cos(\theta_i + \varphi_i) + y_i^{\cdot} \sin(\theta_i + \varphi_i) - \xi_i^{\cdot} \sin\varphi_i + \theta_i^{\cdot} \xi_i \cos\varphi_i$$

and neglecting the terms of the second and higher order of smallness, we obtain the required kinematic equations of motion of the vehicle on pneumatic tire wheels along a curvilinear path

$$x_i^{\cdot} \sin(\theta_i + \varphi_i) - y_i^{\cdot} \cos(\theta_i + \varphi_i) + \xi_i^{\cdot} = 0 \qquad (2.1)$$
$$\theta_i^{\cdot} + \varphi_i^{\cdot} - (\alpha_i \xi_i - \beta_i \varphi_i - \gamma_i \chi_i)\,[x_i^{\cdot} \cos(\theta_i + \varphi_i) + y_i^{\cdot} \sin(\theta_i + \varphi_i)] = 0$$

When the deviations from a rectilinear translation taking place with the velocity $V =$ $=$ const in the Oy direction are small, the above equations reduce to the known Keldysh [2] equations.

Let now $T = T(q, q^\cdot, t)$ be the kinetic energy of the vehicle the position of which is defined by n generalized coordinates q_j ($j = 1, 2, \ldots, n$), $Q_j = Q_j(q, q^\cdot, t)$ be the prescribed generalized forces applied to the system and $R_j = R_j(\xi, \varphi, \chi)$ the generalized forces governed by the deformation of the tires. To find the functions R_j we first compute the virtual work done by the deforming forces

$$\delta A = \sum_{i=1}^{m} [F_i(\delta x_i \sin\theta_i - \delta y_i \cos\theta_i) + M_{\theta_i}\delta\theta_i + M_{\chi_i}\delta\chi_i] =$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m}\left[F_i\left(\frac{\partial x_i}{\partial q_j}\sin\theta_i - \frac{\partial y_i}{\partial q_j}\cos\theta_i\right) + M_{\theta_i}\frac{\partial\theta_i}{\partial q_j} + M_{\chi_i}\frac{\partial\chi_i}{\partial q_j}\right]\delta q_j$$

and from these we obtain

$$R_j = \sum_{i=1}^{m}\left[F_i\left(\frac{\partial x_i}{\partial q_j}\sin\theta_i - \frac{\partial y_i}{\partial q_j}\cos\theta_i\right) + M_{\theta_i}\frac{\partial\theta_i}{\partial q_j} + M_{\chi_i}\frac{\partial\chi_i}{\partial q_j}\right] \qquad (2.2)$$

Here the forces F_i and the moments M_{θ_i} and M_{χ_i} are given by (1.6). Having taken into account all the forces acting on the system including the reactions between the tires and the road, we obtain the required dynamic equations for the vehicle in the usual form

$$\frac{d}{dt}\frac{\partial T}{\partial q_j^\cdot} - \frac{\partial T}{\partial q_j} = Q_j + R_j \qquad (j = 1, 2, \ldots, n) \qquad (2.3)$$

where the generalized forces R_j are given by (2.2). Equations (2.3) together with (2.1) represent the equations of motion along a curvilinear path of a vehicle on pneumatic tire wheels.

3. Stability of peripheral motion. We begin the study of the stability of peripheral motion of a nehicle on pneumatic tire wheels by constructing equations describing its small deviations from the steady state motion.

Let $\theta_i = \theta_i^\circ + \theta_i'$, where θ_i° is the value of the angle θ_i on the unperturbed trajectory and θ_i' is a small deviation of θ_i from θ_i°. We replace the quantities $x_i^\cdot$, $y_i^\cdot$ by u_i, V_i respectively, using the relations

$$x_i^\cdot = V_i \cos_i\theta_i^\circ + u_i \sin\theta_i^\circ, \qquad y_i^\cdot = V_i \sin\theta_i^\circ - u_i\cos\theta_i^\circ \qquad (3.1)$$

where $V_i = \text{const}$ is the longitudinal velocity component of the ith wheel during the peripheral motion of the vehicle and u_i is the transverse displacement velocity of the ith wheel (its magnitude is of the order of the other small quantities). Inserting (3.1) into (2.1) and linearizing with respect to the small quantities we obtain the following kinematic equations for the vehicle on pneumatic tire wheels when the deviations from its peripheral motion are small:

$$\begin{aligned} &u_i + \xi_i^\cdot + V_i\theta_i' + V_i\varphi_i = 0 \\ &\theta_i^\cdot + \varphi_i^\cdot - \alpha_i V_i\xi_i + \beta_i V_i\varphi_i + \gamma_i V_i\chi_i = 0 \end{aligned} \qquad (3.2)$$

The dynamic equations of motion retain, in this case, their form (2.3).

Similarly to the case of the steady state rectilinear motion, the equations describing small deviations from the peripheral motion can be simplified when either the velocity of motion V_i, or the values of the kinematic parameters α_i, β_i and γ_i are large.

The case when the velocities of motion are large. In accordance with the general theory [1] the velocities V_i are assumed large if the following inequalities hold:

$$\tau \gg \tau_i \qquad (i = 1, 2, \ldots, m) \tag{3.3}$$

where τ represents the least duration of the transient processes in the variables q_1, q_2, $\ldots$, q_n, and τ_i is given by

$$\tau_i = 2\,\mathrm{Re}\,[\beta_i V_i (1 + \sqrt{1 - 4\alpha_i / \beta_i^2})]^{-1}$$

When the motion is curvilinear, the velocities V_i should also be bounded from above. This follows from the assumption that the deformations ξ_i, φ_i and χ_i are small.

Assuming that all these conditions hold and carrying out the reasoning analogous to that of [1] we find, that the region of slow motions is determined by

$$\theta_i' + \varphi_i + d\pi_i / ds_i = 0, \qquad \alpha_i \xi_i - \beta_i \varphi_i - \gamma_i \chi_i - d\theta_i / ds_i = 0 \tag{3.4}$$

where π_i is a quasi-coordinate corresponding to the variable $u_i \equiv \pi_i$. Eliminating ξ_i and φ_i from (1.7) and using the relations (3.4) we find

$$F_i = \frac{a_{1i}}{V_i}\theta_i^{\cdot} - a_{2i}\theta_i' - \frac{a_{21}}{V_i}u_i + a_{3i}\chi_i \qquad (i = 1, 2, \ldots, m) \tag{3.5}$$

$$M_{\theta_i} = -\frac{b_i}{V_i}u_i - b_i\theta_i',$$

$$M_{\chi_i} = -\frac{b_{1i}}{V_i}\theta_i^{\cdot} + b_{2i}\theta_i' + \frac{b_{2i}}{V_i}u_i - b_{3i}\chi_i$$

Here the positive coefficients a_{ki} and b_{ki} are connected with the parameters of the pneumatic tire by the relations

$$a_{1i} = \frac{a_i}{\alpha_i}, \qquad a_{2i} = \frac{a_i\beta_i}{\alpha_i}, \qquad a_{3i} = \frac{a_i\gamma_i}{\alpha_i} + \sigma_i N_i \tag{3.6}$$

$$b_{1i} = \frac{\sigma_i N_i}{\alpha_i}, \qquad b_{2i} = \frac{\sigma_i N_i \beta_i}{\alpha_i}, \qquad b_{3i} = N_i\left(\frac{\sigma_i\gamma_i}{\alpha_i} + \rho_i\right)$$

Inserting the expressions (3.5) for the forces and moments into (3.1) we obtain the expressions for R_j.

The case when the kinematic parameters are large. The quantities α_i, β_i and γ_i are assumed to be sufficiently large [1] if the inequalities (3.3) in which τ_i is defined by $\tau_i = (\beta_i V_i)^{-1}$, hold. We introduce a small parameter μ such that the relations

$$\mu\alpha_i = \alpha_i^\circ, \ \mu\beta_i = \beta_i^\circ, \ \mu\gamma_i = \gamma_i^\circ, \ \mu\Omega_i = \Omega_i^\circ \, (\Omega_i \equiv \theta_i^{\circ\cdot})$$

hold. Here α_i°, β_i° and γ_i° are finite quantities and Ω_i° are small quantities of the same order as the small deformations. We write the second group of the kinematic equations (3.2) in the form

$$\mu(\theta_i'' + \varphi_i^{\cdot}) = \alpha_i^\circ V_i \xi_i - \beta_i^\circ V_i \varphi_i - \gamma_i^\circ V_i \chi_i - \Omega_i^\circ$$

With μ sufficiently small, we have a system of differential equations in which the highest derivative is accompanied by a small parameter. In the present case the variables undergoing rapid variation are represented by the sums $\theta_i' + \varphi_i$. When $\mu \to 0$, a surface of slow motions appears in the phase space. This surface is stable with respect to the rapid motions and

$$\varphi_i = \varkappa_{1i}\xi_i - \varkappa_{2i}\chi_i - (\beta_i V_i)^{-1}\Omega_i \qquad (\varkappa_{1i} = \alpha_i/\beta_i, \ \varkappa_{21} = \gamma_i/\beta_i) \tag{3.7}$$

where $\varkappa_{1i}$ and $\varkappa_{2i}$ are the transverse creep coefficients, hold for this surface. Using (3.7) to eliminate φ_i from the first group of the kinematic equations we obtain

$$u_i + \xi_i^{\cdot} + V_i\theta_i' + \varkappa_{1i}V_i\xi_i - \varkappa_{2i}V_i\chi_i - \beta_i^{-1}\Omega_i = 0 \qquad (3.8)$$

and inserting (3.7) into (1.6) we find

$$F_i = a_i\xi_i + \sigma_i N_i\chi_i, \qquad M_{\chi_i} = -\sigma_i N_i\xi_i - \rho_i N_i\chi_i \qquad (3.9)$$

$$M_{\theta_i} = \varkappa_{1i}b_i\xi_i - \varkappa_{2i}b_i\chi_i - b_i(\beta_i V_i)^{-1}\Omega_i$$

Thus in the present case when the kinematic parameters have large values, the equations of motion are represented by (2.3) and (3.8) and expressions (3.9) should be used in computing R_j

Equations (2.2), (2.3) and (3.5) or respectively (2.2), (2.3) and (3.8), can be regarded as an extension of the generalized transverse creep hypothesis [1] to the case of curvilinear motion.

4. Examples. 1. Stability of peripheral rolling of a pneumatic tire wheel. We shall consider the case when the angular velocity ω of the characteristic rotation of the wheel is kept constant, i.e. when an additional (rheonomic) constraint $\omega =$ const is imposed on the motion of the wheel. The Lagrangian function has the form

$$2L = m(x^{\cdot 2} + y^{\cdot 2}) + A(\theta^{\cdot 2} + \chi^{\cdot 2}) + 2\omega C\chi\theta^{\cdot} + rN\chi^2$$

Here $N = mg$ is the weight of the wheel, A, C are the diametral and axial moments of inertia, respectively, x, y are the Cartesian coordinates of the center of the wheel, r is the distance between the center of the wheel and the point K (x_1, y_1), and the coordinates of the latter are given by

$$x_1 = x - r\chi\sin\theta, \quad y_1 = y + r\chi\cos\theta \qquad (4.1)$$

Let the generalized forces be $Q_x = Q_y = Q_\theta = Q_\chi = 0$. In accordance with the formulas (1.6), (2.2) and (4.1) the generalized forces R_j are given by the following expressions

$$R_x = (a\xi + \sigma N\chi)\sin\theta, \quad R_y = -(a\xi + \sigma N\chi)\cos\theta$$
$$R_\theta = b_\varphi, \quad R_\chi = -(ar + \sigma N)\xi - (\rho + r\sigma)N\chi$$

The dynamic equations of motion of the wheel are

$$mx^{\cdot\cdot} - (a\xi + \sigma N\chi)\sin\theta = 0, \qquad my^{\cdot\cdot} + (a\xi + \sigma N\chi)\cos\theta = 0 \qquad (4.2)$$
$$A\theta^{\cdot\cdot} + \omega C\chi^{\cdot} - b\varphi = 0 \qquad A\chi^{\cdot\cdot} - \omega C\theta^{\cdot} + (ar + \sigma N)\xi - (r - r\sigma - \rho)N\chi = 0$$

Using (4.1) let us pass from the variables $x^{\cdot}$ and y' to u and V by means of (3.1). Discarding the quantities of the second and higher order of smallness we find

$$x^{\cdot}\cos\theta^\circ + y^{\cdot}\sin\theta^\circ = V, \quad x^{\cdot}\sin\theta^\circ - y^{\cdot}\cos\theta^\circ = u + r\chi^{\cdot}$$
$$x^{\cdot\cdot}\cos\theta^\circ + y^{\cdot\cdot}\sin\theta^\circ = 0, \quad x^{\cdot\cdot}\sin\theta^\circ - y^{\cdot\cdot}\cos\theta^\circ = u^{\cdot} + r\chi^{\cdot\cdot} - V\Omega$$

where $\Omega = \theta^{\circ\cdot} =$ const is the value of the angular velocity $\theta^{\cdot}$ during the perpheral motion of the wheel. Instead of (4.2) we obtain

$$mu^{\cdot} + mr\chi^{\cdot\cdot} - mV\Omega - a\xi - \sigma N\chi = 0, \quad A\theta^{\cdot\cdot} + \omega C\chi^{\cdot} - b\varphi = 0 \qquad (4.3)$$
$$A\chi^{\cdot\cdot} - \omega C\theta^{\cdot} + (ar + \sigma N)\xi - (r - r\sigma - \rho)N\chi = 0$$

which, together with the kinematic equations

$$u + \xi^{\cdot} + V\theta' + V\varphi = 0, \qquad \Omega + \theta^{\cdot\cdot} + \varphi^{\cdot} - \alpha V\xi + \beta V\varphi + \gamma V\chi = 0 \qquad (4.4)$$

form a complete system of equations for determining u, θ, χ, ξ and φ. When the wheel is rolling along a circle of radius $R = V/\Omega = \omega r/\Omega$, the steady state values u_0, χ_0, ξ_0 and φ_0 $(\theta^\circ = \Omega t)$ satisfy the following equations:

$$mV\Omega + a\xi_0 + \sigma N\chi_0 = 0, \qquad \Omega - \alpha V\xi_0 + \gamma V\chi_0 = 0$$
$$\omega C\Omega - (ar + \sigma N)\xi_0 + (r - r\sigma - \rho)N\chi_0 = 0 \tag{4.5}$$

The conditions of existence of nontrivial solutions of (4.5) is fulfilled when the velocity has a unique value $V = V_0$, where

$$V_0^2 = \frac{gr\,[a(r-\rho) + \sigma^2 N]}{(a\gamma + \alpha\sigma N)(d^2 + r^2) - rN\,[\alpha(r-\rho) - \gamma\sigma]} \tag{4.6}$$

Here d is the radius of inertia of the wheel.

When $V = V_0$, we obtain a set of circular motions of the wheel. The tilt χ_0 of the wheel and the lateral deformation ξ_0 of the pneumatic tire are expressed in terms of the radius $R = V_0 / \Omega$ of the circumscribed circle by means of the relations

$$\xi_0 = \frac{\Omega(\sigma N - m\gamma V_0^2)}{V_0(a\gamma + a\sigma N)}, \qquad \chi_0 = -\frac{\Omega(a + \alpha m V_0^2)}{V_0(a\gamma + \alpha\sigma N)} \tag{4.7}$$

The numerator in (4.6) is always positive. The denominator is positive when $a\gamma > \alpha N(1 - \sigma)$. When $a\gamma < \alpha N(1 - \sigma)$ which may happen when the product αN is large (normally $\sigma \approx 0.5$ to 0.7), the denominator becomes zero when $r = r_*$. Since the theory used here is applicable only when the deformations are small, we ought to limit ourselves to the values $r \ll r_*$. Under these conditions the quantity ξ_0 is (as implied by the assumption of the positiveness of Ω and V_0 and by (4.7)) always positive. The quantity χ_0 is (as follows from (4.7)) always negative. The form assumed by the rolling pneumatic tire wheel is shown on Fig. 2.

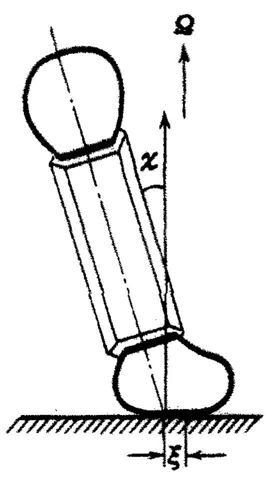

Fig. 2.

To inspect the stability of motion we construct equations describing small deviations of the wheel from its steady state motion. Denoting the small deviations by a prime we obtain from (4.3) - (4.5)

$$mu'^{\cdot} + mr\gamma'^{\cdot} - a\xi' - \sigma N\chi' = 0, \tag{4.8}$$
$$A\theta'^{\cdot\cdot} + \omega C\chi'^{\cdot} - b\varphi' = 0$$
$$A\chi'^{\cdot\cdot} - \omega C\theta'^{\cdot} + (ar + \sigma N)\xi' - (r - r\sigma - \rho)N\chi' = 0$$
$$u' + \xi'^{\cdot} + V\theta' + V\varphi' = 0,$$
$$\theta'^{\cdot} + \varphi'^{\cdot} - \alpha V\xi' + \beta V\varphi' + \gamma V\chi' = 0$$

These equations also describe the small deviations of the wheel from a rectilinear motion, the latter case regarded here as a particular case of the circular motion. The characteristic equation of the system (4.8) can be written as

$$pP(p) = 0 \tag{4.9}$$
$$P(p) = p^6 + \beta Vp^5 + [\alpha_0 + b_0 + (\alpha_1 + 1)\tau]p^4 + \beta V(\alpha_0 + \tau)p^3 +$$
$$+ (\alpha_0 b_0 - \beta_0 + \gamma_1\tau + \alpha_1\tau^2)p^2 + \beta V(\gamma_0\tau - \beta_0)p + \beta_1 b_0(\tau - \tau_0)$$
$$\alpha_0 = a_0 + re - e_1,\ \alpha_1 = \alpha k^{-2},\ \tau = k^2V^2,\ \tau_0 = k^2V_0^2 = \beta_0\beta_1^{-1}$$
$$\beta_0 = a_0 e_1 + g\sigma e,\ \beta_1 = \gamma\gamma_0 k^{-1} + \alpha_1(kg\sigma - e_1)$$
$$\gamma_0 = a_0 + ek^{-1},\ \gamma_1 = a_0 + \alpha_1(b_0 - rkb_0 - e_1) + \gamma k^{-1}(b_0 + ek^{-1})$$
$$a_0 = \frac{a}{m},\ b_0 = \frac{b}{A},\ k = \frac{C}{rA},\ e = \frac{ar + \sigma N}{A},\ e_1 = \frac{N(r - r\sigma - \rho)}{A}$$

The zero root of (4.9) depends on the manifold of the steady state motions of the pneumatic tire wheel. Indeed, the steady state values of the variables in (4.8) satisfy the relations

$$u_0' + V\theta_0' = 0, \qquad \xi_0' = \varphi_0' = \chi_0' = 0$$

and this implies that the steady state motions form a one-dimensional manifold. Its physical meaning is reflected in the fact that the motion of the wheel may establish itself along a straight line in any direction. The stability of the manifold of the rectilinear motions is determined by the roots of the characteristic equation $P(p) = 0$. When $V = V_0$ (i.e. $\tau = \tau_0$), the free term of the polynomial $P(p)$ vanishes. An additional zero root of the characteristic equation appearing under these conditions is due to the fact that when $V = V_0$, a manifold of circular motions of the pneumatic tire wheel is generated. The stability of this manifold is determined by the roots of the following characteristic equation:

$$p^5 + \beta V_0 p^4 + [\alpha_0 + b_0 + (\alpha_1 + 1)\tau_0]p^3 + \beta \dot{V}_0(\alpha_0 + \tau_0)p^2 +$$
$$+ (\alpha_0 b_0 - \beta_0 + \gamma_1\tau_0 + \alpha_1\tau_0^2)p + \beta V_0(\gamma_0\tau_0 - \beta_0) = 0$$

The set of circular motions of a pneumatic tire wheel is stable when the following inequality holds

$$a_1\beta_0^2 + a_2\beta_0\beta_1 + a_3\beta_1^2 + b_0^2\beta_1^3 > 0 \qquad (4.10)$$
$$a_1 = \alpha_1(b_0 + ek^{-1})[\alpha_1(kr - 1) + 1 - \gamma k^{-1}]$$
$$a_2 = (\gamma_1 - \gamma_0)(\alpha_0\alpha_1 + b_0 + \gamma_0 - \gamma_1) + \alpha_1(\alpha_1\beta_0 - 2b_0\gamma_0)$$
$$a_3 = b_0[\alpha_0(\alpha_0\alpha_1 + b_0 + \gamma_0 - \gamma_1) + 2\alpha_1\beta_0 - \gamma_0 b_0]$$

For a wheel with the following parameters:

$N = 100$ kg	$\alpha = 20\,\mathrm{m}^{-2}$	$a = 2\cdot10^4$ kg m^{-1}
$r = 0.5$ m	$\beta = 10\,\mathrm{m}^{-1}$	$b = 10^3$ kg m rad^{-1}
$\rho = 0.1$ m	$\gamma = 1\,\mathrm{m}^{-1}$	$A = 0.5$ kg m sec^2
$\sigma = 0.6$	$m = 10$ kg m^{-1}sec^2	$C = 1$ kg m sec^2

the inequality (4.10) holds and $V_0 = 2.2$ m/sec. Thus the rolling of a pneumatic tire wheel along a circle takes place at a definite velocity $V = V_0$, and the trace of the middle plane of the wheel is always parallel to the tangent to the circle.

For comparison purposes we shall reconsider this problem using the Rocard's transverse creep hypothesis [3]. According to this hypothesis the transverse creep acting on the wheel is accompanied by a transverse opposing force $F = -a_c V^{-1}u - a_c\theta'$, where a_c is the creep resistance coefficient. Using these expressions we arrive at the following equations of motion for a pneumatic tire wheel

$$mu + mr\chi^{\cdot\cdot} - mV\Omega + a_c V^{-1}u + a_c\theta' = 0, \qquad A\theta^{\cdot\cdot} + \omega C\chi^{\cdot} = 0$$
$$A\chi^{\cdot\cdot} - \omega C\theta^{\cdot} - rN\chi - a_c rV^{-1}u - a_c r\theta' = 0$$

When the motion is steady state, the variables u_0 and χ_0 are given by

$$u_0 = \frac{mV^2\Omega}{a_c}, \qquad \chi_0 = -\frac{\Omega V(C + mr^2)}{r^2 N} \qquad (4.11)$$

From this it follows that according to the Rocard hypothesis a pneumatic tire wheel can move along a circle with any velocity V, and the trace of the middle plane of the circle forms a constant angle with the tangent to the circle

$$\varepsilon = \operatorname{arctg}\frac{u_0}{V} = \operatorname{arctg}\frac{mV\Omega}{a_c}$$

The stability of motion of the wheel is determined by the roots of the characteristic equation

$$p^3 + a_c \frac{A + mr^2}{mAV} p^2 + \frac{C^2V^2 - r^3NA}{r^2A^2} p + a_c \frac{C(C + mr^2)V^2 - r^3NA}{mr^2VA} = 0$$

The motion of a pneumatic tire wheel is stable, when the inequality

$$V^2 > r^3NA/C(C - A) \tag{4.12}$$

holds. We see that the Rocard hypothesis gives a result which is intrinsically different from that obtained in accordance with the Keldysh theory. It is interesting to note that the rolling of a perfectly rigid wheel with classical nonholonomic constraints gives exactly the same result. In this case the equations of motion have the form

$$A\theta'' + \omega C\chi' = 0, \qquad (A + mr^2)\chi'' - \omega(C + mr^2)\theta' - rN\chi = 0$$

From this it follows that the angle of inclination χ of the rigid wheel during its motion along a circle is

$$\chi_0 = -\omega\Omega(C + mr^2)/rN$$

which agrees with (4.11). The stability of the manifold of the circular motions of the wheel is determined by the roots of the characteristic equation

$$A(A + mr^2)p^2 + \omega^2C(C + mr^2) - rNA = 0$$

Consequently the rolling of a rigid wheel is conservatively stable when the inequality $V^2 > r^3NA/C(C + mr^2)$ holds, and the latter inequality differs from (4.12) only in unimportant details.

2. Stability of a circular motion of an automobile. We investigate the motion of the simplest model of an automobile on identical pneumatic tire wheels. Let replace the wheels by the equivalent front and the near wheel and let the front wheels be turned to the left by a constant angle ψ. We consider the case of high velocity motion, since in this case we can use (3.5) to calculate the forces and the moments. Using the notation employed in Fig. 3 we obtain the following equations of motion

$$mu_1' + 2a_2 uV^{-1} - ml_1\theta'' - mV\Omega + 2a_2\theta' - cV^{-1}\theta' = -a_2\psi$$

$$c_1V^{-1}u_1 + mk^2\theta'' + c_2V^{-1}\theta' + c_1\theta' = (a_2l_2 - b)\psi$$

$$c = 2a_1 + a_2l, \quad c_1 = a_2(l_1 - l_2) + 2b, \quad c_2 = a_1(l_2 - l_1) + l(a_2l_2 - b)$$

where k is the radius of inertia of the automobile relative to the vertical axis passing through its center of mass. When the automobile moves in a circle, the steady state values u_1° and Ω of the variables u_1 and θ' satisfy the following equations:

$$2a_2u_1^\circ - (c + mV^2)\Omega = -a_2V\psi, \qquad c_1u_1^\circ + c_2\Omega = (a_2l_2 - b)V\psi$$

which in turn yield

$$u_1^0 = MV\psi[a_1(a_2l - 2b) + mV^2(a_2l_2 - b)], \quad \Omega = a_2^2lMV\psi \tag{4.13}$$

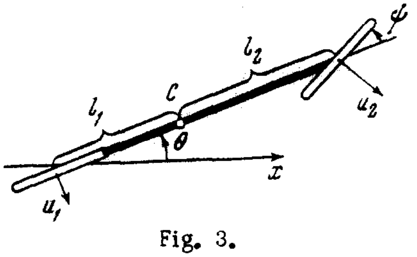

Fig. 3.

Here

$$M = [a_2^2l^2 + 4a_1b + c_1mV^2]^{-1} \tag{4.14}$$

Using (4.13) and the obvious relations $u_2 = u_1 - l\theta' + V\psi$, we readily obtain

$$u_2^\circ = MV\psi[a_1(a_2l + 2b) + mV^2a)_2l_1 + b)]$$

Let $\Omega > 0$ and $V > 0$. Then the second relation of (4.13) implies that the inequality

$M\psi > 0$ must hold. Thus two cases of motion of an automobile in a circle are possible.

(a) $M > 0,\ \psi > 0$ when we have the usual motion illustrated on Fig. 4a.

(b) $M < 0,\ \psi < 0$ when we have the extraordinary motion depicted on Fig. 4b.

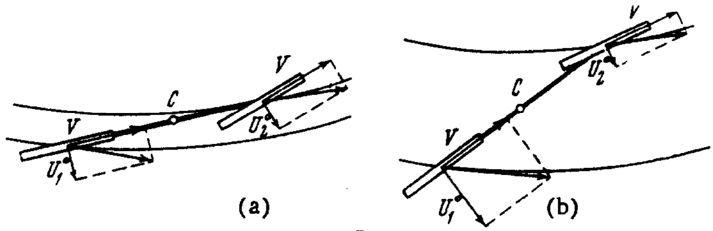

Fig. 4.

In the latter case the front wheels are turned in the direction opposite to that in which the automobile is turning (the possibility of such a motion of an automobile in a circle was found by A. A. Khachaturov in computation using an analog computer). By (4.14), this case arises when the inequalities

$$l_2 > l_1 + \frac{2b}{a_2}, \qquad V^2 > \frac{a_2^2 l^2 + 4a_1 b}{m\,[a_2(l_2 - l_1) - 2b]} \equiv V^2_*$$

are satisfied. The center of mass of the automobile is displaced towards the rear wheels and the velocity of motion exceeds some critical value $V = V_*$.

The stability of the manifold of motions of an automobile along a circle is determined by the roots of the characteristic equation $mk^2p^2 + m\,(2a_2k^2 + c_1l_1 + c_2)\,p + M = 0$. Since the coefficients accompanying p are always positive, the motion of the automobile is stable whei $M > 0$. Consequently the usual mode of motion of an automobile with a blocked steering along a circle is always stable, and the extraordinary mode is always unstable.

In conclusion we note that the study of the motions of the model under consideration using the Rocard transverse creep hypothesis yields the same qualitative result. This can be explained by the fact that, while in the first example the moment of the forces arising from the torsion of the pneumatic tire during the rolling of a single pneumatic tire wheel is significant, in the second example the corresponding moments acting on the front and rear wheels do not influence the dynamics of the automobile to any appreciable extent.

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